

Influence of surrounding neutral medium on the stability of a fluid cylinder carrying an axial current

By G. S. MURTY¹

With 2 figures in the text

ABSTRACT

The influence of a surrounding neutral medium on the stability of a compressible fluid cylinder carrying a uniform axial current is investigated. It is shown that the presence of the neutral medium results in increased effective density of the fluid and thereby leads to reduced growth rate for the hydromagnetic instabilities for the fluid cylinder.

1. Introduction

The possibility of confining a hot plasma by means of magnetic fields is extensively investigated [1] both experimentally and theoretically. Since the charged particles are "tied" to the magnetic field lines, it should in principle be possible to confine them to a region by a suitable choice of the magnetic field configuration. However, the drift motions of the charged particles due to the electric and magnetic fields, the magnetic field gradients and the collective phenomena like the hydromagnetic instabilities set an upper limit on the time in which a plasma can be confined to a given region.

The essential feature of a magnetically confined plasma is that it is separated from the material walls of the container by a region of very low gas pressure.

Recently Alfvén and Smårs [2] pointed out the advantages in surrounding a plasma in the presence of magnetic fields by a neutral gas. In particular they showed that if a quantity of heat Q is given to a cylindrical plasma per unit length by external sources (e.g. by electromagnetic induction) and Q is greater than a certain critical value, Q_0 , one obtains "burn out" of the plasma whose temperature increases to very high values, because the thermal conductivity is greatly reduced due to the magnetic field. The presence of the neutral gas between the plasma and the walls prevents contamination of the plasma with the wall material, a fact which is of considerable advantage in reducing the radiation losses due to bremsstrahlung. If the density of the neutral gas is higher than that of the plasma it is expected that the growth rate of the

¹ On deputation from the Tata Institute of Fundamental Research, Appollo Pier Road, Bombay, India.

hydromagnetic instabilities is considerably reduced from its value when the plasma is surrounded by vacuum. We shall study in this paper a simplified model of the plasma by assuming it is a fluid which is compressible and electrically conducting. It will be shown that there is a significant reduction in the growth rate of the hydromagnetic instabilities when a cylindrical plasma carrying an axial current is surrounded by a relatively high density neutral medium

2. Formulation of the problem

In the experiments reported by Alfvén and Smårs [2] and Smårs and Johansson [3], a glass torus with a mean diameter of 10 cm and a bore diameter of about 3 cm is filled with helium gas at pressures ranging from 1 to 400 torr. The torus of gas, which is made electrically conducting by means of a pre-ionisation current, forms the secondary winding of a two-turn transformer through which a condenser bank of $2.2 \mu F$ capacity charged to a voltage of 30 kV is discharged. When the discharge is induced inside the high pressure gas, a relatively small region carries the whole current and this region is surrounded by a high density neutral medium. Our aim is to study the stability of the conducting gas when it carries the electric current and is surrounded by the neutral medium. A rigorous theory of the hydromagnetic stability of a toroidal configuration presents enormous difficulties and therefore we shall only study the case of a cylindrical configuration which is simpler to treat analytically.

To be more specific, we shall imagine an infinitely long cylinder of radius R of a compressible fluid of small electrical conductivity carrying a uniform axial current which is assumed to be produced by sources outside the fluid. Let the electrical conductivity be so small that the magnetic field is not frozen into the medium. (The reason for this assumption will be apparent later.) The conducting fluid is assumed to be surrounded by another compressible fluid which is an ideal insulator so that no current can leak into it. Let the non-conducting fluid extend up to radial distance ΛR where $\Lambda > 1$. Let a cylindrical rigid coaxial non-conducting wall bound the non-conducting fluid.

The configuration will be in equilibrium with the following distributions of j_e the current density, B_e the magnetic induction and p_e the pressure. (Whenever we use the word pressure we shall always understand hereafter the gas pressure only.)

$$j_e = \begin{cases} (0, 0, j_0), & \text{for } 0 \leq r \leq R, \\ 0, & \text{for } R \leq r \leq \Lambda R, \end{cases} \quad (2.1)$$

$$B_e = \begin{cases} (0, \frac{1}{2} \mu_0 j_0 r, 0) & \text{for } 0 \leq r \leq R, \\ (0, \frac{1}{2} \mu_0 j_0 R^2/r, 0) & \text{for } R \leq r \leq \infty, \end{cases} \quad (2.2)$$

$$p_e = \begin{cases} p_2 = \frac{1}{2} \mu_0 j_0^2 (R^2 - r^2) & \text{for } 0 \leq r \leq R, \\ p_2 & \text{for } R \leq r \leq \Lambda R. \end{cases} \quad (2.3)$$

In eqs. (2.1), (2.2) and (2.3) the suffix e of j_e , B_e and p_e denotes that these quantities refer to the equilibrium situation; p_2 denotes in eq. (2.3) the pressure in the non-conducting medium. Since the fluid is assumed to be compressible

the density is not uniform if the temperature is uniform in the conducting fluid in which the electric current is flowing. If we assume that the conducting and non-conducting fluids have uniform temperatures T_1 and T_2 respectively and that the equation of state in the simple form

$$p = n \kappa T, \quad (2.4)$$

where n is the *total* particle density, κ is the Boltzmann constant and T is the temperature holds, then the distribution of the density, ϱ_e , in the equilibrium is given by

$$\varrho_e = \begin{cases} \varrho^* = \frac{m_i}{2 \kappa T_1} (p_2 + \frac{1}{4} \mu_0 j_0^2 (R^2 - r^2)), & \text{for } 0 \leq r \leq R, \\ \varrho_2 = \frac{m}{\kappa T_2} p_2, & \text{for } R \leq r \leq \Lambda R, \end{cases} \quad (2.5)$$

where m_i is the mass of the ion in the conducting medium and m is the mass of the neutral atom in the surrounding non-conducting medium. In eq. (2.5) it is assumed that the conducting gas is composed of only electrons and singly ionised atoms and that the medium is electrically neutral on the average. If we assume that the nuclei of the ions in the conducting region and of the atoms in the non-conducting region are the same, that is, if the plasma is surrounded by the atomic gas itself, we have $m_i \approx m$. Also, in eq. (2.5) the terms proportional to the ratio of the electron mass to the ion mass are neglected and it is assumed that the electron and the ion temperatures are the same.

3. Basic equations of the problem and the approximations

If we ignore the transport phenomena like the thermal conduction and the viscous dissipation the basic equations of the problem are, in rationalised M.K.S. units,

$$\varrho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B}, \quad (3.1)$$

$$\frac{d\varrho}{dt} = -\varrho \nabla \cdot \mathbf{v}, \quad (3.2)$$

$$\frac{d}{dt} (p \varrho^{-\gamma}) = 0, \quad (3.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (3.4)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.5)$$

$$\text{and} \quad I = \text{constant}. \quad (3.6)$$

Eq. (3.1) is the equation of motion without the terms representing the viscous dissipation; eq. (3.2) expresses the conservation of mass; eq. (3.3) shows that the changes in the pressure and the density are adiabatic, γ being the ratio of the two specific heats; eqs. (3.4) and (3.5) are two of Maxwell's equations (the remaining not being written as they are not required for our purpose) and eq. (3.6) expresses a constraint that the total current, I , flowing in the conducting fluid cylinder is a constant and is maintained by sources outside the fluid. The set of eqs. (3.1) to (3.5) are the usual equations employed in the study of the behaviour of a hydromagnetic system. In the case when the fluid has a finite electrical conductivity an additional equation determining the rate of change of induction, namely,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}, \quad (3.7)$$

where σ is the electrical conductivity of the fluid has to be added to the above set of equations. However, in the present problem we assume that the electrical conductivity of the fluid is so small that the time scale, τ_d , of the diffusion of the magnetic field lines is much smaller than any other time scale of the problem. The relevant time scale with which we can compare τ_d is the time scale of the hydrodynamic instabilities, which we may call τ_h . Our assumption implies that

$$\tau_d / \tau_h \ll 1. \quad (3.8)$$

It can be seen from the considerations of order of magnitude of various terms in eq. (3.7) that, in view of the inequality (3.8), the second term on the right hand side of eq. (3.7) is the dominant term and that the remaining terms can be neglected in comparison with it. We can therefore assume that

$$\nabla^2 \mathbf{B} = 0, \quad (3.9)$$

where the right-hand side can differ from zero by terms of the order of the ratio τ_d / τ_h . Using eqs. (3.4), (3.5) and (3.9) we find that

$$\nabla \times \mathbf{j} = 0. \quad (3.10)$$

Eq. (3.10), together with the constraint expressed by eq. (3.6) and the boundary condition that the current density vector has no component normal to the interface between the conducting and the non-conducting fluids, enables one to determine the current density distribution inside the conducting fluid uniquely.

Before preceeding ahead we remark here that the growth rate of the hydromagnetic instabilities for an incompressible fluid cylinder carrying an axial current obtained by Murty [4, 5] on the basis of the approximations of the type mentioned above do not differ very much from those obtained by Tayler [6] on the assumption of infinitely large electrical conductivity for the fluid, provided the magnetic field is produced only by the uniformly distributed electric current flowing inside the fluid itself. In view of this fact, we shall continue to work with the above-mentioned approximation also in the case of an ionised gas which is assumed to be a compressible fluid whose electrical conductivity

may not be necessarily small in reality. One has, however, an advantage in such a procedure which allows one to obtain the electromagnetic field distribution from Maxwell's equations with suitable boundary conditions without regard to the fluid velocity and thereby leads to some analytical simplification.

In order to study the stability of the equilibrium we shall assume that the interface between the conducting and the non-conducting fluids is deformed due to an arbitrarily small perturbation into a shape given in circular cylindrical coordinates with z -axis coinciding with the axis of the fluid cylinder by

$$r = R + a \cos \varphi, \quad (3.11)$$

where
$$\varphi \equiv -m\theta + kz, \quad (3.12)$$

where m is an integer and k is a positive real number, and a depends on time such that

$$a = a_0 \exp(\omega t). \quad (3.13)$$

In eq. (3.12) $k = 2\pi/\lambda$ is the wave number, λ being the wavelength of the perturbation in the z -direction. The amplitude of the perturbation is assumed to be small to start with so that

$$a/R \ll 1.$$

In calculating any physical quantity we shall ignore terms that are proportional to the second and the higher powers of a . Further we shall only consider the cases when ω is a real number and ignore the cases when ω is a complex quantity. Our aim is to find the dispersion relation between ω and k . For any given m and k and other physical quantities if ω is found to have positive real value, we conclude that the interface between the conducting and the non-conducting fluids is unstable and therefore the equilibrium we have started with is unstable for arbitrarily small perturbations of the fluid interface. We shall not consider those unstable perturbations that may exist and keep the fluid interface unperturbed since these are physically less important in the stability considerations than those that do disturb the fluid interface.

When the fluid interface is disturbed the time-dependent part of all the physical quantities is in the first approximation proportional to the quantity a . The distribution of the current density and the magnetic field has been discussed by Murty [5] in connection with a similar stability problem where the fluid cylinder was surrounded by vacuum. Since the electromagnetic quantities are determined, entirely by Maxwell's equations (eqs. (3.4) and (3.5)) and the boundary conditions at the boundary of the conducting and the non-conducting fluids and independently of the fluid velocities, the presence of a non-conducting medium surrounding the conducting medium is immaterial as far as their distributions after the perturbation are concerned. In the following paragraph we can assume that the perturbed distributions of $\mathbf{j}$ and $\mathbf{B}$ in eq. (3.1) are known and are the same as those obtained in [5].

If we take the divergence on both sides of eq. (3.1) and note that the fluid was in static equilibrium before the perturbation and use eqs. (3.4) and (3.10) we obtain, neglecting higher order terms,

$$\varrho \frac{\partial}{\partial t} \nabla \cdot \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \cdot \nabla \varrho = -\nabla^2 p - \mu_0 j^2. \quad (3.14)$$

Using eqs. (3.2) and (3.3), eq. (3.14) can be rewritten

$$-\varrho \frac{\partial}{\partial t} \left(\frac{1}{\gamma p} \frac{dp}{dt} \right) + \frac{\partial \mathbf{v}}{\partial t} \cdot \nabla \varrho = -\nabla^2 p - \mu_0 j^2. \quad (3.15)$$

In eq. (3.15) the first term on the left-hand side is

$$-\varrho \frac{\partial}{\partial t} \left(\frac{1}{\gamma p} \frac{dp}{dt} \right) = -\frac{\partial}{\partial t} \left(\frac{\varrho}{\gamma p} \frac{dp}{dt} \right) + \frac{1}{\gamma p} \frac{dp}{dt} \frac{\partial \varrho}{\partial t}, \quad (3.16)$$

where the second term on the right-hand side is of a higher order than the first and so can be neglected in the linear approximation. Therefore, eq. (3.15) now becomes, after some rearrangement of the terms,

$$-\frac{\partial}{\partial t} \left(\frac{1}{C_s^2} \frac{\partial p}{\partial t} \right) + \frac{\partial}{\partial t} \left\{ \mathbf{v} \cdot \left(-\frac{\nabla p}{C_s^2} + \nabla \varrho \right) \right\} = -\nabla^2 p - \mu_0 j^2, \quad (3.17)$$

where we have further neglected one higher order term and

$$C_s = \left(\frac{\gamma p}{\varrho} \right)^{\frac{1}{2}} \quad (3.18)$$

is the acoustic speed. In the conducting medium the acoustic speed, C_{s1} , is given by

$$C_{s1} = \left(\gamma_1 \frac{2\kappa T_1}{m_i} \right)^{\frac{1}{2}}. \quad (3.18a)$$

In the linear approximation we have only to use the zero order (equilibrium) distributions of the pressure and the density as given by eqs. (2.3) and (2.5) in order to evaluate the second term on the left hand side of eq. (3.17). Using eqs. (2.3) and (2.5) we find that in the conducting fluid

$$\left(-\frac{\nabla p}{C_s^2} + \nabla \varrho \right) = \left[\frac{1}{2} \mu_0 j^2 r \left(\frac{1}{C_{s1}^2} - \frac{m_i}{2\kappa T_1} \right) \right] \hat{\mathbf{r}}, \quad (3.19)$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction. We see from eq. (3.19) that the terms involving the gradients of the pressure and the density on the left hand side of eq. (3.17) drop out if $C_{s1}^2 = 2\kappa T_1/m_i$ which, however, is not true because C_{s1} is given by eq. (3.18a). The second term on the left hand side of eq. (3.17) can be shown, under typical conditions of the experiments at high pressure gas discharges, to be smaller than the second term on the right hand side of the same equation. In order to do so let us consider the ratio, α ,

$$\alpha \equiv \frac{\frac{\partial}{\partial t} \left\{ \mathbf{v} \cdot \left(-\frac{\nabla p}{C_s^2} + \nabla \varrho \right) \right\}}{\mu_0 j^2} \sim \frac{|\omega|^2 L^2 \left(\frac{\gamma-1}{\gamma} \right)}{4 \frac{\kappa T_1}{m_i}}, \quad (3.20)$$

where L is the characteristic linear dimension of the problem and $|\omega|$ is the characteristic growth rate of the instabilities. For a calculation of the order of magnitude of the above ratio, the values of the various quantities from the experiments mentioned by Alfvén and Smårs [2] and Smårs and Johansson [3] can be taken as representative. In these experiments $L \approx 4 \times 10^{-3}$ m; $m_i \approx 9 \times 10^{-27}$ kg (helium); $T_1 \approx 10^5$ °K; the current density $\approx 3 \times 10^8$ Am $^{-2}$; the mass density $\approx 4 \cdot 10^{-8}$ kg m $^{-3}$ (corresponding to a pre-discharge pressure of 400 torr); the value of $|\omega|$ corresponding to the above mentioned values of the current density and the mass density is, according to the results of the present theory, approximately 10^6 s $^{-1}$. When these values are used the ratio, α , turns out to be approximately equal to 10^{-2} . We can therefore simplify the eq. (3.17) as

$$\left(-\frac{1}{C_{s1}^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) p^i = -\mu_0 j^2, \quad (3.21)$$

for the conducting medium. In the non-conducting medium the pressure, p^e , is given by

$$\left(-\frac{1}{C_{s2}^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) p^e = 0, \quad (3.21 \text{ a})$$

which follows from eq. (3.17) without any further approximations as the current density is zero in this medium. The acoustic speed in the non-conducting medium is given by

$$C_{s2} = \left(\gamma_2 \frac{\kappa T_2}{m} \right)^{\frac{1}{2}}. \quad (3.22)$$

The superscripts i and e for p in eqs. (3.21) and (3.21 a) are used to distinguish the pressure in the conducting and the non-conducting media respectively.

4. Pressure distribution

In eq. (3.21) the current density distribution is given [5] up to first order terms by

$$\mathbf{j} = (j_r, j_\theta, j_z), \quad (4.1 \text{ a})$$

$$\left. \begin{aligned} \text{where} \quad j_r &= -a k j_0 \frac{I'_m(kr)}{I'_m(kR)} \sin \varphi, \\ j_\theta &= \frac{a m j_0}{r} \frac{I_m(kr)}{I'_m(kR)} \cos \varphi, \\ \text{and} \quad j_z &= j_0 - a k j_0 \frac{I_m(kr)}{I'_m(kR)} \cos \varphi, \end{aligned} \right\} \quad (4.1)$$

when $0 \leq r \leq R + a \cos \varphi$ and is identically zero when $R + a \cos \varphi \leq r \leq \Lambda R$. If we use eqs. (4.1) in eq. (3.21) we get

$$\left(-\frac{1}{C_{s1}^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right) p^i = -\mu_0 j_0^2 + 2\mu_0 a k j_0^2 \frac{I_m(kr)}{I'_m(kR)} \cos \varphi. \quad (4.2)$$

We shall solve eq. (4.2) by expanding p^i as a power series in the parameter a which characterises the perturbation and write

$$p^i = p_0^i(r) + a p_1^i(r, \theta, z), \quad (4.3)$$

where the higher order terms are neglected. If we substitute for p^i from eq. (4.3) in eq. (4.2) and separate terms of the corresponding orders we obtain

$$\nabla^2 p_0^i(r) = -\mu_0 j_0^2, \quad (4.4)$$

and
$$\left(-\frac{1}{C_{s1}^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right) a p_1^i(r, \theta, z) = 2\mu_0 a k j_0^2 \frac{I_m(kr)}{I'_m(kR)} \cos \varphi. \quad (4.5)$$

Since a was assumed to be given by eq. (3.13), eq. (4.5) becomes

$$\left(-\frac{\omega^2}{C_{s1}^2} + \nabla^2\right) p_1^i(r, \theta, z) = 2\mu_0 k j_0^2 \frac{I_m(kr)}{I'_m(kR)} \cos \varphi, \quad (4.6)$$

which suggests that $p_1^i(r, \theta, z)$ can be separated in the variables as

$$p_1^i(r, \theta, z) = p_1^i(r) \cos \varphi, \quad (4.7)$$

so that $p_1^i(r)$ satisfies the equation

$$\left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr}\right) - \left(l^2 + \frac{m^2}{r^2}\right)\right] p_1^i(r) = 2\mu_0 k j_0^2 \frac{I_m(kr)}{I'_m(kR)}, \quad (4.8)$$

where

$$l^2 \equiv \left(\frac{\omega^2}{C_{s1}^2} + k^2\right). \quad (4.9)$$

It can be shown that the general solution of the inhomogeneous ordinary differential eq. (4.8) is given by

$$p_1^i(r) = A_1 I_m(lr) + \frac{2\mu_0 k j_0^2}{k^2 - l^2} \frac{I_m(kr)}{I'_m(kR)}, \quad (4.10)$$

where A_1 is an arbitrary constant and $I_m(lr)$ is the modified Bessel function (regular at the origin) in whose argument the quantity l is defined as the positive square root of the right-hand side of eq. (4.9). In the non-conducting fluid the radial part of the first order pressure distribution is given by

$$p_1^e(r) = D_1 I_m(hr) + D_2 K_m(hr), \quad (4.11)$$

where D_1 and D_2 are arbitrary constants, $K_m(hr)$ is the modified Bessel function (regular at infinity) and the quantity h in the argument is defined as the positive square root of

$$h^2 \equiv \left(\frac{\omega^2}{C_{s2}^2} + k^2 \right). \quad (4.12)$$

We close this section by writing the solution of eq. (4.4):

$$p_0^i(r) = E - \frac{1}{4} \mu_0 j_0^2 r^2, \quad (4.13)$$

where E is an arbitrary constant.

5. Boundary conditions and the dispersion relation

We assumed that the non-conducting medium is bounded at $r = \Lambda R$ by a rigid coaxial cylindrical wall and therefore one boundary condition is that the radial component of the fluid velocity is zero there. Since there are no electromagnetic forces in this medium, this condition is equivalent to the vanishing of the radial derivative of the first order pressure $r = \Lambda R$, which means (see eq. (4.11)) that

$$D_1 I'_m(\Lambda h R) + D_2 K'_m(\Lambda h R) = 0, \quad (5.1)$$

where the prime to the Bessel functions denotes differentiation with respect to the argument. The continuity of the pressure at the deformed boundary given by eq. (3.11) gives another condition. The pressure in the conducting fluid, p^i , is given by eqs. (4.3), (4.7), (4.10) and (4.13):

$$p^i = E - \frac{1}{4} \mu_0 j_0^2 r^2 + a \left\{ A_1 I_m(lR) + \frac{2\mu_0 k j_0^2}{k^2 - l^2} \frac{I_m(kr)}{I'_m(kR)} \right\} \cos \varphi. \quad (5.2)$$

The pressure in the non-conducting fluid is given by

$$p^e = F + a \{ D_1 I_m(hr) + D_2 K_m(hr) \} \cos \varphi, \quad (5.3)$$

which is the general solution of eq. (3.21 a), F being an arbitrary constant. The continuity of the pressure at $r = R + a \cos \varphi$ means that p^i equal p^e there. Equating the terms of the corresponding order we obtain

$$E - \frac{1}{4} \mu_0 j_0^2 R^2 = F, \quad (5.4)$$

$$\text{and} \quad -\frac{1}{2} \mu_0 j_0^2 R + \frac{2\mu_0 k j_0^2}{k^2 - l^2} \frac{I_m(kR)}{I'_m(kR)} + A_1 I_m(lR) = D_1 I_m(hR) + D_2 K_m(hR), \quad (5.5)$$

where we have neglected higher order terms proportional to a^2 . In eq. (5.4) F can be identified with p_2 in eqs. (2.5) and therefore E is determined. Here-

after we need consider only eq. (5.5) in connection with obtaining the dispersion relation.

In addition to the two eqs. (5.1) and (5.5) we can obtain as shown in the following two more equations for the four unknowns ω^2 , A_1 , D_1 and D_2 . By virtue of eqs. (3.11) and (3.13), eq. (3.1) can be rewritten as

$$\varrho_b a \omega^2 \cos \varphi = \left\{ -\frac{\partial p}{\partial r} + (\mathbf{j} \times \mathbf{B})_r \right\}_b, \quad (5.6)$$

where the suffix 'b' to the brace and in ϱ_b indicates that the respective quantities must be evaluated at boundary approached from either side. The only quantity in eq. (5.6) that is not listed so far is the distribution of $\mathbf{B}$. As was mentioned in §3 we can write down at once the distribution of $\mathbf{B}$ in the present problem from [5] where it is calculated for a case when there is an infinitely conducting coaxial sheath at some distance from the conducting fluid. For the present calculations we need only the distribution of $\mathbf{B}$ in the absence of such a conducting sheath and it is given by (note the change in units)

$$\mathbf{B} = (B_r, B_\theta, B_z), \quad (5.7 a)$$

$$\text{where} \quad \left. \begin{aligned} B_r &= \left\{ A I'_m(kr) + \frac{\mu_0 a m j_0}{I'_m(kR)} \frac{I_m(kr)}{kr} \right\} \sin \varphi, \\ B_\theta &= \frac{1}{2} \mu_0 j_0 r - \left\{ \mu_0 a j_0 \frac{I'_m(kr)}{I'_m(kR)} + m A \frac{I_m(kr)}{kr} \right\} \cos \varphi, \end{aligned} \right\} \quad (5.7)$$

$$\text{and} \quad B_z = A I_m(kr) \cos \varphi,$$

$$\text{where} \quad A = -\mu_0 a m j_0 \frac{I_m(kR) K_m(kR)}{I'_m(kR)}, \quad (5.7 b)$$

in the region $0 \leq r \leq R + a \cos \varphi$. We shall not write the distribution of $\mathbf{B}$ in the outer region as it is not required for our purpose. If we substitute for ϱ^* , $\mathbf{j}$, p^i and $\mathbf{B}$ from eqs. (2.5), (4.1), (5.2) and (5.7) respectively in eq. (5.6), we obtain, using eq. (5.7 b)

$$\varrho_1 \omega^2 = -A_1 l I'_m(lR) + \mu_0 j_0^2 \left[1 + \frac{2k^2}{l^2 - k^2} + \frac{kR}{2} \frac{I_m(kR)}{I'_m(kR)} - \frac{m^2}{kR} \frac{I_m(kR)}{I'_m(kR)} I_m(kR) K_m(kR) \right], \quad (5.8)$$

$$\text{where} \quad \varrho_1 \equiv (\varrho^*)_{r=R} = \frac{m_i p_2}{2 \kappa T_2}. \quad (5.9)$$

Similarly if we substitute for p^e from eq. (5.3) and $\varrho = \varrho_2$, the density of the outer medium in eq. (5.6) we obtain

$$\varrho_2 \omega^2 = -D_1 h I'_m(hR) - D_2 h K'_m(hR). \quad (5.10)$$

Eliminating the constants A_1 , D_1 and D_2 among the four eqs. (5.1), (5.5), (5.8) and (5.10) we obtain

$$\omega^2 [\varrho_1 + \varrho_2 G(m, \Lambda, lR, hR)] = \mu_0 j_0^2 F(m, kR, lR), \quad (5.11)$$

where

$$\begin{aligned} F(m, kR, lR) \equiv & 1 + \frac{2k^2}{l^2 - k^2} + \frac{2kl}{k^2 - l^2} \frac{I_m(kR)}{I'_m(kR)} \frac{I'_m(lR)}{I_m(lR)} \\ & + \frac{kR I_m(kR)}{2 I'_m(kR)} - \frac{lR I'_m(lR)}{2 I_m(lR)} \\ & - \frac{m^2 I_m(kR)}{kR I'_m(kR)} I_m(kR) K_m(kR), \end{aligned} \quad (5.12)$$

and

$$G(m, \Lambda, lR, hR) \equiv \frac{l I'_m(lR)}{h I_m(lR)} \Psi(m, \Lambda, hR), \quad (5.13)$$

where

$$\Psi(m, \Lambda, hR) \equiv \frac{I_m(hR) K'_m(\Lambda hR) - K_m(hR) I'_m(\Lambda hR)}{I'_m(\Lambda hR) K'_m(hR) - K'_m(\Lambda hR) I'_m(hR)}. \quad (5.14)$$

Eq. (5.11) is the required dispersion relation which determines the characteristic values of ω for given values of the remaining parameters.

6. Discussion

We shall discuss eq. (5.11) in the following limiting cases.

(a) Both fluids incompressible

If we assume that both the conducting and the non-conducting fluids are incompressible, that is, C_{s1} and C_{s2} have infinitely large values, we can show that eq. (5.11) reduces to

$$\omega^2 [\varrho_1 + \varrho_2 Q(m, \Lambda, kR)] = \mu_0 j_0^2 P(m, kR), \quad (6.1)$$

where

$$Q(m, \Lambda, kR) \equiv \frac{I'_m(kR)}{I_m(kR)} \frac{I_m(kR) K'_m(\Lambda kR) - I'_m(\Lambda kR) K_m(kR)}{I'_m(\Lambda kR) K'_m(kR) - K'_m(\Lambda kR) I'_m(kR)}, \quad (6.2)$$

and

$$\begin{aligned} P(m, kR) \equiv & 1 + \frac{kR}{2} \left\{ \frac{I'_m(kR)}{I_m(kR)} - \frac{I_m(kR)}{I'_m(kR)} \right\} \\ & - \frac{m^2 I_m(kR)}{kR I'_m(kR)} \{1 + I_m(kR) K_m(kR)\}. \end{aligned} \quad (6.3)$$

If we further assume that the conducting fluid is surrounded by vacuum, that is, if $\varrho_2 = 0$ in eq. (6.1), the resulting equation¹ is the same as the one derived and discussed in detail in [4, 5] where it was shown that $\omega^2 \geq 0$ for values of

¹ In this connection see also the appendix at the end of this paper.

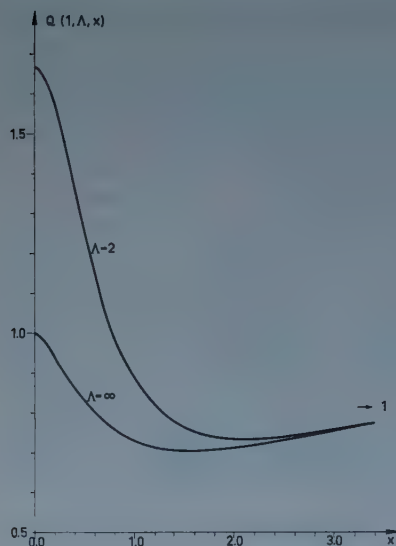


Fig. 1.

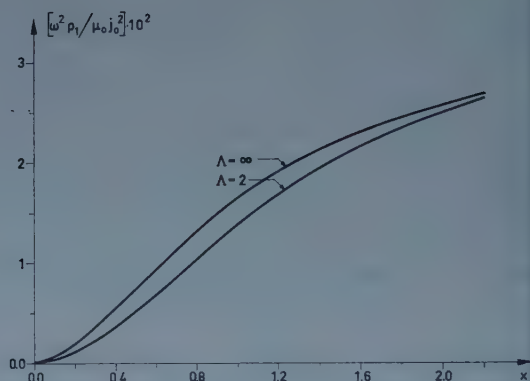


Fig. 2.

Fig. 1. Variation of $Q(1, \Lambda, x)$ with $x = kR$ for $\Lambda = 2$ and ∞ . The function Q is a measure of the effective density of the incompressible and non-conducting fluid surrounding the incompressible and conducting fluid cylinder for a perturbation of a given wavelength $\lambda = 2\pi/k$; R is the radius of the conducting fluid cylinder. The value 1 in the arguments of the function Q indicates that Q corresponds to the helical perturbations of the fluid cylinder. Λ is the ratio of the radius of the rigid and non-conducting coaxial cylinder to R . As x tends to infinitely large values, the value of $Q(1, \Lambda, x)$ tends to 1 for both $\Lambda = 2$ and ∞ .

Fig. 2. Variation of ω^2 (in non-dimensional units) as given by eq. (6.1) for helical perturbations with $x = kR$, for two values of Λ , the ratio of the radius of the rigid and non-conducting coaxial cylindrical wall to R . The influence of this cylindrical wall is negligible, particularly for perturbations of the fluid cylinder with very small values of the wavelength along the axis. The ratio ρ_2/ρ_1 is assumed to be 10.

$kR \geq 0$ (which is the only range of the values of kR which is of interest to us) when $|m| = 0$ or 1. It was also shown there that when $|m| > 1$ there exists a range of values of kR from zero up to a certain value, $(kR)_c$, such that $\omega^2 \geq 0$ according as $kR \geq (kR)_c$. Since $\omega^2 > 0$ corresponds to the unstable modes of the perturbation the fluid cylinder was shown to be unstable for all the values of m and the growth rate was shown to tend to the same maximum value for all the values of m as the value of kR tends to infinitely large values.

If $\rho_2 \neq 0$, only the left hand side of eq. (6.1) is effected while the right hand side is unaltered. The function $Q(1, \Lambda, kR)$, which may be considered as a measure of the effective density of the non-conducting medium for a perturbation of a given wavelength, is plotted versus kR in Fig. 1 for two values of Λ , namely, 2 and ∞ . It is seen in this figure that the function Q is positive for all the positive values of kR and that when $kR > 3$ the value of $Q(1, 2, kR)$ is nearly the same as that of $Q(1, \infty, kR)$. Both the functions will approach the limiting value 1 as kR tends to infinitely large values. For the same values

of Λ , namely, 2 and ∞ , the quantity $(\omega^2 \varrho_1 / \mu_0 j_0^2)$ as given by eq. (6.1) is plotted versus kR in Fig. 2 assuming $\varrho_2 / \varrho_1 = 10$, a value which is suggested by the calculations of Fälthammar [7]. It is seen in Fig. 2 that the quantity ω^2 is not very sensitive to the variation of Λ . In fact when kR tends to infinitely large values ω^2 tends to the same maximum value for all the values of m and for all the values of $\Lambda > 1$. This maximum value depends on ϱ_1 , ϱ_2 and j_0 and is given by

$$|\omega| = \left[\frac{\mu_0 j_0^2}{2(\varrho_1 + \varrho_2)} \right]^{\frac{1}{2}}. \quad (6.4)$$

which shows that the growth rate of the hydromagnetic instabilities of an incompressible fluid cylinder carrying an axial current is significantly reduced when the fluid cylinder is surrounded by a non-conducting and incompressible fluid of relatively high density.

(b) *Incompressible conducting fluid surrounded by compressible non-conducting fluid*

In the high pressure gas discharges where the current carrying channel is surrounded by a non-conducting gas, the temperature of the conducting gas is larger than that of the non-conducting gas by a factor of hundred or even more and therefore the conducting gas has a correspondingly higher acoustic speed than that in the non-conducting gas. Under such conditions an approximate value of the growth rate of the instability can be obtained by taking the ideal case of an incompressible conducting fluid surrounded by a non-conducting compressible fluid. The dispersion relation under these conditions can be shown to be given by

$$\omega^2 \left[\varrho_1 + \varrho_2 \frac{k I'_m(kR)}{h I_m(kR)} \Psi(m, \Lambda, hR) \right] = \mu_0 j_0^2 P(m, kR), \quad (6.5)$$

which is obtained from eq. (5.11) by letting C_{s1} tend to infinitely large values.

A complete solution of eq. (6.5) for ω as a function of kR for given values of the remaining parameters is not simple because the equation is an implicit equation for ω , but we notice that both sides of the equation involve only real functions of real arguments if ω is a real quantity which shows that there can be real values of ω which are solutions of eq. (6.5). Among the values of ω , those corresponding to large values of k are of particular interest since the instabilities have in general larger growth rates, the larger the value of k is. An examination of eq. (6.5) shows that ω^2 cannot have infinitely large value for any value of kR . Therefore we see that

$$\omega^2 / k^2 C_{s2}^2 \ll 1,$$

when $kR \rightarrow \infty$, that is, ω tends to a finite limit when k tends to infinitely large values. If we expand all the functions in eq. (6.5) in a power series in $\omega^2 / k^2 C_{s2}^2$ and $1/kR$ and retain only the leading terms, we find that ω^2 has a

finite value plus higher order terms which vanish in the limit of infinitely large values of k . The limiting value of $|\omega|$ thus obtained will be found to be the same as that when both the fluids are assumed to be incompressible. This means that the growth rate of perturbations of infinitesimally small wavelength in the present case also is given by eq. (6.4).

(c) *Both the fluids compressible*

If both the conducting and the non-conducting fluids are assumed to be compressible and the non-conducting fluid is bounded by a rigid coaxial cylindrical wall, the dispersion relation is given by eq. (5.11) where both the quantities l and h occur. We see that this dispersion relation also involves only functions that are real functions of a real argument if ω is a real quantity. Following the arguments that are employed in the previous paragraph, one can conclude that the limiting value of ω^2 obtained in this case when kR tends to infinitely large values is also the same as the one obtained when both the fluids are assumed to be incompressible and that the growth rate of the perturbations of infinitesimally small wavelengths is given by eq. (6.4).

7. Concluding remarks

We have shown that a conducting fluid cylinder carrying an axial electric current and surrounded by a non-conducting fluid which is bounded at some distance from the axis is less unstable than when it is surrounded only by vacuum, the remaining conditions being the same in the two situations. The influence of the surrounding neutral medium is manifested only in the increased effective density of the conducting fluid. The effects due to the compressibility of the fluids can be ignored for perturbations of infinitesimally small wavelength. The growth rate of the instabilities has a maximum value for perturbations of vanishingly small wavelength.

An important physical factor that should be taken into account when the wavelength of the perturbation becomes very small is the viscosity of the fluids. Even though it is very difficult to estimate the viscosity of the ionised gas in the presence of an electric current, its non-zero value, however small it may be, will lead to viscous stresses that cannot be ignored for perturbations of very small wavelengths. As we have already seen, the perturbations of vanishingly small wavelengths have the largest growth rate in the absence of viscous forces. Due to the damping effects introduced by the viscous forces one can expect that the growth rate of the unstable modes should tend to zero for perturbations of vanishingly small wavelength [6]. This gives rise to the result that there is a wavelength of the perturbation for which the perturbation grows at maximum rate in the presence of viscous forces. We can obtain an idea of the approximate value of the wavelength of the perturbation that grows faster than the perturbations with other values of the wavelength from the calculations of Tayler. He showed that a compressible and ideally conducting fluid cylinder with surface current distribution and surrounded only by vacuum has a maximum growth rate corresponding to a perturbation of a certain wavelength, λ_m , whose approximate value is given by

$$\lambda_m = 6 R (d/R)^{\frac{1}{2}}, \quad (7.1)$$

where d is the mean free path of the particles in the compressible fluid. The value of λ_m is in general smaller than R since $d \ll R$, a condition which is implicitly assumed to be valid when the fluid approximation is made in stability calculations. The value of the wavelength of the perturbation that grows at a maximum rate in the present problem also can reasonably be expected, although we have not proved it, to be of the same order as λ_m given above. Therefore, the growth rate of the perturbations of infinitesimally small wavelength, given by eq. (6.4), can be taken to be a first approximation to the actual growth rate when viscous forces are taken into account. We can only say from our previous calculations that the growth rate obtained for perturbations of vanishingly small wavelength is an upper limit and the actual growth rate is smaller than the one given by eq. (6.4).

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I am very thankful to Professor H. Alfvén for suggesting this problem and for his continued encouragement and to Mr. E. Åström for useful discussions.

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Department of Electronics, Royal Institute of Technology, Stockholm 70.

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Note added in the proof: In the references (4) and (5) given in the present paper the name "Taylor" should be read, whenever it occurs, as "Tayler".

APPENDIX

In [4, 5] Murty has obtained a dispersion relation when $\varrho_2 = 0$ taking into account the influence of the surface tension of the fluid. It can be shown that, when $\varrho_2 \neq 0$, the required dispersion relation for the case of incompressible fluids including the effect of interfacial tension can be obtained by adding to the right hand side of eq. (6.1) of the present paper a term

$$-\frac{T k R I'_m(k R)}{R^3 I_m(k R)} (k^2 R^2 + m^2 - 1),$$

where T is the interfacial tension between the conducting and the non-conducting fluids.

Very recently, there have appeared in print two independent papers: Breus, S. N., J. Tech. Phys. (in Russian) 30, 1010 (1960), and Tayler, R. J., Rev. Mod. Phys. 32, 908 (1960). These authors have considered the problem of stability of twisted magnetic fields in a fluid of finite electrical conductivity of an arbitrary value. They obtain also, as a limiting case, a dispersion relation when the fluid has a vanishingly small electrical conductivity. This dispersion relation is the same, except for formal differences, as the one obtained by Murty [4, 5] if the surface tension of the fluid is put equal to zero in his calculations.

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